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The Polya Algorithm in L_∞ Approximation

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1. INTRODUCTION

The so-called Polya algorithm is the construction of a best L_∞ approximation as the limit of unique best L_p approximations as $p \to \infty$. This limit is known to exist in a number of situations, and in each case, the limit function is a best L_∞ approximation which is better than the others in some way. There are also examples in which the Polya algorithm fails to converge. See [1, Theorem 1.1; 2, Theorem 1; 8, Sects. 1–5; 9, Sects. 12–7; 10, Theorem 1].

In this article, we consider the Polya algorithm in two quite different settings. In the first, we consider the problem of approximating functions in $L_\infty(I)$ by non-decreasing functions. We show that, in general, the algorithm fails to converge a.e. by constructing a bounded, Lebesgue measurable function h on $[0, 2]$ such that $\limsup h_p(x) > \liminf h_p(x)$ for $x \in [1, 2]$.

The second setting involves approximating functions in $L_\infty(\Omega, \mathcal{A}, \mu; X)$ by functions in $L_\infty(\Omega, \mathscr{B}, \mu; X)$, where $(\Omega, \mathcal{A}, \mu)$ is a probability space, $\mathscr{B}$ is sub-σ-algebra of $\mathscr{A}$, and X is a uniformly convex Banach space. If $X = R$, the Polya algorithm converges. This was shown by Darst in [2]. In fact, if

$$f_\infty(x) = \lim_{p \to \infty} f_p(x),$$

where f_p is the best L_p approximation to f, then $f_\infty|_E$ is a best L_∞ approximation to f_E for each $E \in \mathscr{B}$. This property makes f_∞ a uniquely best best L_∞ approximation to f in this setting. If X is an arbitrary uniformly convex Banach space, the proof used in [2] can be appropriately modified by using Chebyshev centers and diameters.

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2. Monotone Approximation in $L_\infty(I)$

We show that the Polya algorithm in general fails to converge for functions $h \in L_\infty(I)$, where I is a finite interval. For convenience we have $I = [0, 2]$. Specifically, we construct a bounded Lebesgue measurable function $h(x)$ on $[0, 2]$ and sequences (p_k) and (q_k) tending to ∞ so that if h_p is the best L_p-approximate to h by non-decreasing functions, then for some $\varepsilon > 0$,

$$h_{p_k}(x) > h(x) + 4\varepsilon \qquad \text{for} \quad x \in [1, 2]$$

and

$$h_{q_k}(x) < h(x) + 2\varepsilon \qquad \text{for} \quad x \in [1, 2]$$

for sufficiently large k. Clearly then $h_p(x)$ does not converge as $p \to \infty$ for any $x \in [1, 2]$.

Let $p_k = 2^{2k}$ and $q_k = 3^{2^k}$. Define

$$x_{p_k} = 1 - [(2/3)^{p_k} + (1/7)^{p_k}]$$

and

$$y_{q_k} = 1 - (2/3)^{q_k}.$$

We list several properties of (x_{p_k}) and (y_{q_k}) in the following lemma, which we state without proof.

Lemma 1. (i) $x_{p_k} < y_{q_k} < x_{p_{k+1}}$.

(ii) $x_{p_k} \to 1$ *and* $y_{q_k} \to 1$ *as* $k \to \infty$.

(iii) $1 - y_{q_k} = o[(1/7)^{p_k}]$ *as* $k \to \infty$.

(iv) $1 - x_{p_{k+1}} = o(1 - y_{q_k})$ *as* $k \to \infty$.

Now for $k = 1, 2, \ldots$, *define the intervals*

$$A_k = [x_{p_k}, x_{p_k} + (2/3)^{p_k}],$$
$$B_k = (x_{p_k} + (2/3)^{p_k}, y_{q_k}),$$
$$C_k = [y_{q_k}, y_{q_k} + (1/2)(2/3)^{q_k}],$$

and

$$D_k = (y_{q_k} + (1/2)(2/3)^{q_k}, x_{p_{k+1}}).$$

Let

$$A = \bigcup_{k=1}^{\infty} (A_k \cup C_k)$$

and

$$B = \bigcup_{k=1}^{\infty} (B_k \cup D_k).$$

Now define $h(x)$ *by*

$$\begin{aligned} h(x) &= 8 \qquad \text{if} \quad x \in [0, x_{p_1}] \cup A \\ &= 0 \qquad \text{if} \quad x \in B \\ &= 6 \qquad \text{if} \quad x \in [1, 2]. \end{aligned}$$

LEMMA 2. *Let* $0 < \varepsilon < 1/8$ *be fixed and let* h_p *denote the best* L_p-*approximate to* h *by non-decreasing functions. Then for sufficiently large* k,

$$h_{p_k}(x) > 6 + 4\varepsilon \qquad \text{for} \quad x \in [1, 2].$$

Proof. If not, then for some arbitrarily large values of k,

$$h_{p_k}(x) \leqslant 6 + 4\varepsilon \qquad \text{for} \quad x \in [0, 1].$$

Define

$$\begin{aligned} h^*_{p_k}(x) &= h_{p_k}(x) \qquad \text{if} \quad x \in [0, x_{p_k}] \\ &= 6 + 8\varepsilon \qquad \text{if} \quad x \in (x_{p_k}, 2). \end{aligned}$$

Since $|h(x) - h_{p_k}(x)| \geqslant 2 - 4\varepsilon$ for $x \in A_k$ we have

$$\begin{aligned} D &= \int_0^2 |h - h_{p_k}|^{p_k}\, d\mu - \int_0^2 |h - h^*_{p_k}|^{p_k}\, d\mu \\ &\geqslant (2 - 4\varepsilon)^{p_k} (2/3)^{p_k} - [(2 - 8\varepsilon)^{p_k} (1 + o(1))(2/3)^{p_k} \\ &\quad + (6 + 8\varepsilon)^{p_k} (1/7)^{p_k} + (8\varepsilon)^{p_k}] \\ &= (2 - 4\varepsilon)^{p_k} (2/3)^{p_k} \left[1 - \left(\frac{2 - 8\varepsilon}{2 - 4\varepsilon}\right)^{p_k} (1 + o(1))\right] + o(1). \end{aligned}$$

Thus for sufficiently large k, $D > 0$, which contradicts the definition of h_{p_k}.

LEMMA 3. *Let* ε *and* h_p *be as in Lemma* 2. *For sufficiently large* k,

$$h_{q_k}(x) < 6 + 2\varepsilon \qquad \text{for} \quad x \in [1, 2).$$

Proof. If not, then let

$$z_k = \inf\{x \in [0, 2]: h_{q_k}(x) > 6 + 2\varepsilon\}$$

and

$$w_k = \inf\{x \in [0, 2]: h_{q_k}(x) > 6 + \varepsilon\}$$

and define

$$\begin{aligned} h^*_{q_k}(x) &= h_{q_k}(x) \qquad \text{if} \quad x \in [0, w_k) \\ &= 6 + \varepsilon \qquad \text{if} \quad x \in [w_k, 2]. \end{aligned}$$

Also let

$$D = \int_0^2 |h - h_{q_k}|^{q_k}\, d\mu - \int_0^2 |h - h^*_{q_k}|^{q_k}\, d\mu.$$

We now consider four cases, depending on the location of z_k.

Case 1. If $z_k \in [1, 2)$, then a better L_{q_k}-approximate can be obtained by lowering $h_{q_k}(x)$ to $6 + 2\varepsilon$ on $[z_k, 2]$, yielding a contradiction.

Case 2. If $z_k \in [y_{q_k}, x_{p_{k+1}})$, then $z_k = y_{q_k}$. For if $z_k \in (y_{q_k} + (1/2)(2/3)^{q_k}, x_{p_{k+1}})$, then a better L_{q_k}-approximate can be obtained by lowering $h_{q_k}(x)$ to $6 + 2\varepsilon$ for $x \in [z_k, x_{p_{k+1}}]$, and if $z_k \in (y_{q_k}, y_{q_k} + (1/2)(2/3)^{q_k})$, then a better L_{q_k}-approximate can be obtained by raising $h_{q_k}(x)$ to $h_{q_k}(y_{q_k} + (1/2)(2/3)^{q_k}$ for $x \in [y_{q_k}, y_{q_k} + (1/2)(2/3)^{q_k}]$. Hence $z_k = y_{q_k}$. Now since $|h(x) - h_{q_k}(x)| \geqslant 6 + 2\varepsilon$ for $x \in (y_{q_k} + (1/2)(2/3)^{q_k}, x_{p_{k+1}})$ and $\int_{w_k}^{q_k} (|h - h_{q_k}|^{q_k} - |h - h^*_{q_k}|^{q_k})\, d\mu \geqslant -(2 - \varepsilon)^{q_k}$, we have

$$\begin{aligned} D &\geqslant (6 + 2\varepsilon)^{q_k} (1 - o(1))(1/2)(2/3)^{q_k} \\ &\quad - [(2 - \varepsilon)^{q_k} (1/2)(2/3)^{q_k} + (6 + \varepsilon)^{q_k} (1 - o(1))(1/2)(2/3)^{q_k} \\ &\quad + \varepsilon^{q_k} + (2 - \varepsilon)^{q_k}] \\ &= (6 + 2\varepsilon)^{q_k} (1/2)(2/3)^{q_k} \left[1 - o(1) - \left(\frac{2 - \varepsilon}{6 + 2\varepsilon}\right)^{q_k} - \left(\frac{6 + \varepsilon}{6 + 2\varepsilon}\right)^{q_k} (1 - o(1)) \right. \\ &\quad \left. + \left(\frac{2 - \varepsilon}{6 + 2\varepsilon}\right)^{q_k} 2(3/2)^{q_k} \right] + o(1). \end{aligned}$$

Thus for sufficiently large k, D is positive, and so $h^*_{q_k}$ would be a better L_{q_k}-approximate.

Case 3. If $z_k \in [0, y_{q_k})$, then an argument similar to that at the beginning of case 2 shows that $z_k \leqslant x_{p_k}$. Since $|h_{q_k}^*(x) - h(x)| \leqslant 6 + \varepsilon$ for $x > w_k$ and $|h_{q_k}(x) - h(x)| > 6 + 2\varepsilon$ for $x \in (x_{p_k} + (2/3)^{p_k}, y_{q_k})$, we have

$$D \geqslant (6 + 2\varepsilon)^{q_k} (1 - o(1))(1/7)^{p_k} - 2(6 + \varepsilon)^{q_k}$$
$$= (6 + 2\varepsilon)^{q_k} (1/7)^{p_k} \left[1 - o(1) - 2\left(\frac{6 + \varepsilon}{6 + 2\varepsilon}\right)^{q_k} 7^{p_k}\right].$$

Clearly $(6 + 2\varepsilon)^{q_k} (1/7)^{p_k} \to \infty$ as $k \to \infty$ and $(6 + \varepsilon/6 + 2\varepsilon)^{q_k} 7^{p_k} \to 0$ as $k \to \infty$. Thus for sufficiently large k, D is positive, and $h_{q_k}^*$ would be a better L_{q_k}-approximate.

Case 4. If $z_k \in [x_{p_{k+1}}, 1)$, then using the fact that h_{q_k} is constant on intervals of the form (x_{p_l}, y_{q_l}) and $y_{q_l}, x_{p_{l+1}})$ and using techniques similar to those used in Cases 2 and 3, it can be shown that

$$\int_{w_k}^{z_k} (|h - h_{q_k}|^{q_k} - |h - h_{q_k}^*|^{q_k})\, d\mu > -[o((2\varepsilon)^{q_k})].$$

Then, since $|h(x) - h_{q_k}(x)| > 2\varepsilon$ for $x \in (1, 2)$, and $|h(x) - h_{q_k}^*(x)| \leqslant 6 + \varepsilon$ for $x \in [z_k, 1)$, we have

$$D \geqslant (2\varepsilon)^{q_k} - [o((2\varepsilon)^{q_k}) + (6 + \varepsilon)^{q_k} 2(2/3)^{p_{k+1}} + \varepsilon^{q_k}].$$

Since $p_{k+1} = 2^{2^{k+1}} = 4^{2^k}$ and $q_k = 3^{2^k}$, we have $(6 + \varepsilon)^{q_k} (2)(2/3)^{p_{k+1}} \to 0$ as $k \to \infty$ so rapidly that D is positive for sufficiently large k. Hence $h_{q_k}^*$ would be a better L_{q_k}-approximate.

We have thus proved the following theorem:

THEOREM 4. *There exists a bounded Lebesgue measurable function $h(x)$ defined on $[0, 2]$ such that $\lim_{p \to \infty} h_p(x)$ does not exist for $x \in [1, 2]$. Hence the Polya algorithm in general fails to converge in $L_\infty(I)$, where I is a finite interval.*

Remarks. It is easy to show that if $h(x)$ is a two-valued function on I, then $\lim_{p \to \infty} h_p(x)$ exists a.e. and equals the average of the two values. The function h in the example has three values, and hence in some sense is a minimal counterexample.

This example shows that two nice results concerning the convergence of the Polya algorithm do not generalize to this case. The first is the result of Darst and Sahab, [3], that the algorithm converges if $h(x)$ is quasi-continuous. The second is the result of Darst, [2], that the algorithm converges if h is $\mathscr{A}$-measurable and h_p must be $\mathscr{B}$-measurable, where $\mathscr{B}$ is a sub-σ-algebra of $\mathscr{A}$. This example shows that $\mathscr{B}$ cannot be taken to be a sub-σ-lattice of $\mathscr{A}$.

3. Vector Valued Functions

In this section we discuss the best best L_∞ approximation to a vector-valued function. The method of proof used in [2] is adapted to this situation by using Chebyshev diameters and centers.

Definition. Let S be a bounded subset of a normed vector space. The Chebyshev radius of S is defined by

$$r(S) = \inf\{\rho: S \subseteq B(\rho, x) \text{ for some } x\}$$

and x_0 is a Chebyshev center of S if

$$S \subseteq \bar{B}(r(S), x_0).$$

The Chebyshev diameter of S is $d(S) = 2r(S)$. It is known that if X is a uniformly convex Banach space, then every non-empty bounded subset S of X has a unique Chebyshev center, denoted by $c(S)$. The Chebyshev radius and center satisfy the following continuity property:

Given $\varepsilon > 0$, there exists $\gamma > 0$ such that if S and T are contained in the unit ball of X and the Hausdorff distance $D(S, T) < \gamma$, then

$$|d(S) - d(T)| < \varepsilon \qquad \text{and} \qquad \|c(S) - c(T)\| < \varepsilon. \tag{$*$}$$

See [5, Section 33].

Let $(\Omega, \mathscr{A}, \mu)$ be a probability space, and let $\mathscr{B}$ be a sub-σ-algebra of $\mathscr{A}$. If X is a uniformly convex Banach space, let $A = L_\infty(\Omega, \mathscr{A}, \mu; X)$ and $B = L_\infty(\Omega, \mathscr{B}, \mu; X)$. See [4, Chap. 4] for a discussion of these spaces. If $f \in A$, let f_p be the best approximate to f in L_p norm by elements of B. We may assume $\|f(x)\| < 1$ for all $x \in \Omega$, and hence also that $\|f_p(x)\| < 1$ for all $x \in \Omega$ and all $p > 1$.

Theorem 2. *$\lim_{p\to\infty} f_p(x)$ exists a.e.*

Actually, since there are uncountably many real numbers $p > 1$ and any f_p can be changed on a set of measure zero, we cannot guarantee that a single exceptional set of measure zero exists in Theorem 2. We must interpret the conclusion of Theorem 2 to mean

There exists a single function $f_\infty(x)$ such that if $\{p_i\}$ is any sequence of real numbers satisfying $1 < p_i$ and $\lim_{i\to\infty} p_i = \infty$, then $\lim_{i\to\infty} f_{p_i}(x) = f_\infty(x)$ a.e.

This interpretation of convergence must also be made in Darst's original paper [2]. Since we can interlace any two sequences, it is enough to prove

$$\lim_{i\to\infty} f_{p_i}(x) \qquad \text{exists a.e.,}$$

where $\{p_i\}$ is a fixed sequence satisfying $\lim_{i\to\infty} p_i = \infty$. Henceforth when we refer to a number p, we mean $p \in \{p_i\}$ and so we can omit the subscripts.

Before proving this theorem, we need some preliminary results. Recall that a vector $x \in X$ is in the essential range of f if

$$\mu(f^{-1}(\mathcal{O})) > 0$$

for every open neighborhood $\mathcal{O}$ of x. We let $f(E)$ denote the essential range of $f|_E$.

The following fact about uniformly convex spaces will be very useful in what follows:

If $R > 1$ is fixed and $\varepsilon > 0$, there is a $\gamma > 0$ such that if $s + \gamma < R$ and $\|x - y\| > \varepsilon$, then $r(\bar{B}(s+\gamma, x) \cap \bar{B}(s+\gamma, y)) < s - \gamma$ $(**)$

Hence if $\varepsilon > 0$, let $\gamma(\varepsilon)$ be a number satisfying $\gamma(\varepsilon) < \varepsilon$, $(*)$ and $(**)$.

Definition. For any $G \in \mathcal{A}$ and $\varepsilon > 0$, we say that $\{S_1, \ldots, S_n\}$ is an ε-antipodal system (for G) if for each i,

(a) $S_i \subseteq G$,

(b) $\mu(S_i) > 0$,

(c) $d(f(S_i)) < \gamma(\varepsilon)/4$,

(d) $d(f(G)) - d(f(\cup S_i)) < \gamma(\varepsilon)$.

The following lemmas show that an ε-antipodal system exists for all G with $\mu(G) > 0$ and $\varepsilon > 0$, and n depends only on ε.

Lemma 3. *Let $E_n \subseteq E_{n+1}$, $E_0 = \bigcup E_n$, $r_n = r(f(E_n))$, and $x_n = c(f(E_n))$ for $n = 0, 1, 2, \ldots$. Then $r_n \to r_0$ and $x_n \to x_0$ as $n \to \infty$.*

Proof. We have $r_n \nearrow \bar{r} \leqslant r_0$.
For $n < m$ we have

$$f(E_n) \subseteq \bar{B}(x_n, r_n)$$

and

$$f(E_n) \subseteq f(E_m) \subseteq \bar{B}(x_m, r_m).$$

Hence

$$f(E_n) \subseteq \bar{B}(x_n, r_n) \cap \bar{B}(x_m, r_m).$$

This implies that $\{x_n\}$ is a Cauchy sequence, since if not, there would exist $\varepsilon > 0$ such that $\|x_n - x_m\| > \varepsilon$ for some arbitrarily large n and m with $n < m$. It would follow from $(**)$ that

$$r_n \leqslant r(\bar{B}(x_n, r_n) \cap \bar{B}(x_m, r_m)) < r_m - \gamma(\varepsilon)/2$$

if n and m are large enough. This is a contradiction, and so $\{x_n\}$ is a Cauchy sequence.

Let $\lim_{n\to\infty} x_n = \bar{x}$. Then if $\varepsilon > 0$,

$$f(E_n) \subseteq B(\bar{x}, \bar{r} + \varepsilon)$$

for large n, and hence

$$f(E_0) \subseteq \bar{B}(\bar{x}, \bar{r} + \varepsilon).$$

It follows that

$$f(E_0) \subseteq \bar{B}(\bar{x}, \bar{r}).$$

This implies $\bar{r} \geqslant r_0$, and hence $\bar{r} = r_0$. It follows that $\bar{x} = x_0$.

LEMMA 4. *Let $E_n \subseteq E_0$ with $\mu(E_n) \to \mu(E_0)$. Define r_n and x_n as in Lemma 3. Then $r_n \to r_0$ and $x_n \to x_0$ as $n \to \infty$.*

Proof. Every function f is the uniform limit of countably valued functions. It follows from $(*)$ that we need only prove this lemma for countably valued functions. Let

$$g = \sum_i a_i \chi_{G_i} \qquad \text{(finite or infinite sum)}$$

and label the sets such that $\mu(G_i) \geqslant \mu(G_{i+1}) > 0$ for all i. Since $\mu(E_n) \to \mu(E_0)$ and since g is constant on each G_i, we have for each k,

$$g\left(\bigcup_{i=1}^{k} G_i\right) \subseteq g(E_n)$$

for sufficiently large n. By Lemma 3, $r(g(\bigcup_{i=1}^{k} G_i)) \to r_0$ as $k \to \infty$. Hence $r_n = r(g(E_n)) \to r_0$ as $n \to \infty$. Property $(**)$ now implies $x_n = c(g(E_n)) \to x_0$ as $n \to \infty$.

LEMMA 5. *Given $G \in \mathcal{A}$ with $\mu(G) > 0$ and $\varepsilon > 0$, there exists an ε-antipodal system $\{S_1, \ldots, S_n\}$ for G. The choice of n depends only on ε.*

Proof. Choose $0 < \delta < \varepsilon$ such that $2\gamma(\delta) < \gamma(\varepsilon)$ and then choose $0 < \eta < \delta/2$ such $3\gamma(\eta) < \gamma(\delta)$. There exists a sequence of simple functions $\{f_k\}$ converging to f in measure. By Lemma 4, there is an index k such that

$$|f_k - f| < \gamma(\gamma(\eta))$$

except on $E_k \subseteq G$, where $\mu(E_k)$ is so small that

$$|d_k - d| < \gamma(\delta),$$

where $d_k = d(f(G \backslash E_k))$ and $d = d(f(G))$. Clearly an η-antipodal system $\{S_1, ..., S_n\}$ exists for f_k for the set $G \backslash E_k$. By the way η and δ were chosen, it may be verified that $\{S_1, ..., S_n\}$ is a δ-antipodal system for f for the set $G \backslash E_k$ and then that $\{S_1, ..., S_n\}$ is an ε-antipodal system for f for the set G.

We now begin to prove Theorem 2. Define the oscillation of f on E to be

$$O(f, E) = d(f(E)).$$

Let P be the set of all countable partitions π of Ω by sets in $\mathscr{B}$. For $h > 0$ and $\pi \in P$, let

$$\delta(h, \pi) = \sum \mu(E),$$

where the sum is over all sets $E \in \pi$ satisfying $O(f, E) \geqslant h$. Let

$$\delta_h = \inf\{\delta(h, \pi) \colon \pi \in \mathscr{P}\}.$$

LEMMA 6. $\delta_h = \delta(h, \pi)$ *for some* $\pi \in \mathscr{P}$.

Proof. Same as in [2, Lemma 2].

Now if

$$E_\pi^h = \bigcup \{E \colon E \in \pi, O(f, E) \geqslant h\},$$

then E_π^h is uniquely determined up to sets of measure zero by the equation $\delta(h, \pi) = \delta_h$. Hence we denote E_π^h by E_h if $\delta(h, \pi) = \delta_h$. Also, if $h_1 < h_2$, then $\mu(E_{h_2} \backslash E_{h_1}) = 0$.

LEMMA 7. *Let* $\varepsilon > 0$. *Choose* σ *so that* $\sigma + \gamma(\sigma) < \gamma(\varepsilon)$. *Let* $h_1, h_2 > 0$ *with* $h_2 - h_1 \leqslant \sigma$. *Let* $F \in \mathscr{B}$, $F \subseteq E_{h_1} \backslash E_{h_2}$. *Then for all* $\alpha > 0$, *there exists* $\beta > 0$ *such that if* $H \in \mathscr{B}$, $H \subseteq F$ *and* $\mu(H) \geqslant \alpha$, *then there exists an* ε-*antipodal system* $\{S_1, ..., S_n\}$ *for* F *such that*

$$\mu(H \cap S_i) \geqslant \beta \mu(H)$$

for all $i = 1, ..., n$.

Proof. Suppose not. Then there exists $\alpha > 0$ and $H_k \subseteq F$ with $\mu(H_k) \geqslant \alpha$ such that for any ε-antipodal system for F,

$$\mu(H_k \cap S_i) < 2^{-k}\mu(H_k)$$

for some i, all k. We may assume

$$\mu(H_k \cap S_1) < 2^{-k}\mu(H_k)$$

for all k. Let

$$H_0 = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} H_k$$

Then $\mu(H_0) \geqslant \alpha$. Let $\{S_1,..., S_n\}$ be a σ-antipodal system for H_0. Since $h_1 \leqslant O(f, H_0) \leqslant h_2$, it may be verified that $\{S_1,..., S_n\}$ is an ε-antipodal system for F. But since

$$\mu(H_k \cap S_1) < 2^{-k}\mu(H_k) < 2^{-k}$$

for all k, we have

$$\mu(H_0 \cap S_1) = \mu(S_1) = 0,$$

a contradiction.

Now let

$$D_p(x) = d(\{f_q(x)\}_{q \geqslant p})$$

and

$$D(x) = \lim_{p \to \infty} D_p(x).$$

Proof of Theorem 2. Let

$$E = \bigcup_{h>0} E_h.$$

We show that $D(x) = 0$ a.e. on E. Since $f_p(x) = f(x)$ a.e. on $\Omega \backslash E$, this will prove that $\lim_{p \to \infty} f_p(x)$ exists a.e.

Let $\varepsilon > 0$. It suffices to show that

$$\mu(x\colon D(x) \geqslant 4\varepsilon) < 2\varepsilon.$$

Choose $\sigma(\varepsilon)$ small enough so that Lemma 7 holds and so that $\mu(E \backslash E_\sigma) < \varepsilon$. Write

$$\begin{aligned} E &= (E \backslash E_\sigma) \cup (E_\sigma \backslash E_{2\sigma}) \cup (E_{2\sigma} \backslash E_{3\sigma}) \cdots \\ &\equiv F_0 \cup F_1 \cup F_2 \cdots . \end{aligned}$$

Let $\{G_{pj}\}_{j=1}^{\infty}$ be a partition of $f_p(E)$ such that $d(G_{pj}) < \gamma(\varepsilon)/4$. Let

$$H_{pji} = f_p^{-1}(G_{pj}) \cap F_i$$

for $i = 1, 2, \ldots$.

Note that $\sum_{i,j} \mu(H_{pji}) < \varepsilon$, where the sum is taken over all indices i and j such that $\mu(H_{pji}) < \varepsilon \cdot 2^{-j-i}$. Hence we consider H_{pji}, where $\mu(H_{pji}) > \varepsilon \cdot 2^{-j-i}$. Let β correspond to $\alpha = \varepsilon \cdot 2^{-j-i}$ as in Lemma 7. Let $m = c(f(F_i))$ and $h = O(f, F_i)$. We complete the proof by showing

$$f_p(H_{pji}) \subseteq B(2\varepsilon, m) \tag{2.1}$$

for sufficiently large p. Suppose this is not true. Then, since

$$d(f_p(H_{pji})) < \gamma(\varepsilon)/4 < \varepsilon,$$

we have that $f_p(H_{pji})$ lies entirely outside of $B(\varepsilon, m)$. Let $y = c(f_p(H_{pji}))$. Then $\|m - y\| \geqslant \varepsilon$. Now let $\{S_1, \ldots, S_n\}$ be an ε-antipodal system for F_i such that

$$\mu(H_{pji} \cap S_k) \geqslant \beta\mu(H_{pji})$$

for all k. Some $f(S_k)$ meets the complement of $B(h/2 + \gamma(\varepsilon), y)$, since if not we would have

$$\bigcup S_k \subseteq \bar{B}(h/2 + \gamma(\varepsilon), y) \cap \bar{B}(h/2 + \gamma(\varepsilon), m)$$

and consequently by $(**)$

$$d\left(\bigcup S_k\right) < h - 2\gamma(\varepsilon),$$

which would contradict the definition of ε-antipodal system. Since $d(f(S_k)) < \gamma(\varepsilon)/4$, it follows that

$$f(S_k) \subseteq \mathscr{C}(B(h/2 + 3\gamma/4, y)).$$

Hence

$$\int_{H_{pji}} \|f - f_p\|^p \, d\mu \geqslant (h/2 + \gamma(\varepsilon)/2)^p \, \mu(H_{pji} \cap S_k)$$
$$> (h/2 + \gamma(\varepsilon)/2)^p \, \alpha\beta.$$

On the other hand, if we define f_p^* to equal f_p off of H_{pji} and m on H_{pji}, then

$$\int_{H_{pji}} \|f - f_p^*\|^p \, d\mu \leqslant (h/2)^p.$$

Hence f_p^* is a better L_p-approximate to f than f_p if p is chosen so that

$$(h/2 + \gamma(\varepsilon)/2)^p \, \alpha\beta > (h/2)^p.$$

This is a contradiction, and (2.1) is verified. The proof of Theorem 2 is completed.

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